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Exactness of G-sequences and monomorphisms ☆

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Abstract

In this paper, we first show monomorphicity of the inclusion map of a CW-pair implies exactness of the G-sequence of the pair. Next we apply exactness of G-sequences to solve monomorphicity of maps, that is, if $n \neq 7$, any map from the n-sphere to S^7 is not monomorphic. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Gottlieb [3] introduced the *Gottlieb group*, $G_n(X)$, of a space X which consists of all $\alpha \in \pi_n(X, x_0)$ such that there exists an affiliated map $A : S^n \times X \to X$ such that $[A|_{S^n \times x_0}] = [\alpha]$ and $A|_{s_0 \times X} = id_X$, where s_0 and x_0 are base points of S^n and X, respectively. This group, $G_n(X)$, is also characterized by

$$G_n(X) = w_\sharp\big(\pi_n(X^X, id_X)\big) \subset \pi_n(X, x_0),$$

where $\omega : X^X \to X$ is an evaluation map at $x_0 \in X$. Thus $G_n(X)$ is also called an *evaluation subgroup* of $\pi_n(X, x_0)$. Gottlieb extensively studied $G_1(X)$ in [2] and $G_n(X)$ for $n \geqslant 2$ in

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[3]. Among other things he has shown that if X is an H-space, then $G_n(X) = \pi_n(X)$ for all n. He also had computed

$$G_n(S^n) = \begin{cases} 0 & \text{for } n \text{ even}, \\ Z & \text{for } n = 1, 3, 7, \\ 2Z & \text{for other odd } n. \end{cases}$$

In [5–7], the Gottlieb groups were generalized by Lee, Kim and Woo as the notions of generalized evaluation subgroups and relative evaluation subgroups. As the homotopy sequence of a topological pair plays an important role in computing homotopy groups, Lee and Woo introduced the G-sequence of a pair which is consisted by subgroups of homotopy groups, that is, Gottlieb groups, generalized evaluation subgroups and relative evaluation subgroups.

Here we introduce the G-sequence of a CW-pair from [6,7]. For convenience, from here on we assume a space is a homotopy type of a CW-complex and a topological pair is a pointed CW-pair.

Let (X, A) be a CW-pair and X^A (or A^A) be the space of all maps from A into X (or from A into A). Then the inclusion map $i : A \to X$ induces the inclusion map $\bar{i} : A^A \to X^A$ given by $\bar{i}(f) = if$. Let $\omega : (X^A, i) \to (X, x_0)$ and $\omega : (X^A, A^A, id) \to (X, A, x_0)$ be the corresponding evaluation maps at the base point $x_0 \in A \subset X$. Then these induce homomorphisms

$$\omega_\sharp : \pi_n(X^A, i) \to \pi_n(X, x_0) \quad \text{and} \quad \omega_\sharp : \pi_n(X^A, A^A, id) \to \pi_n(X, A, x_0).$$

The generalized evaluation subgroups $G_n(X, A)$ are defined by $\omega_\sharp(\pi_n(X^A, i)) = \{[f] \in \pi_n(X) \mid \exists \text{ map } H : A \times I^n \to X \text{ such that } [H|_{x_0 \times I^n}] = [f] \text{ and } H|_{A \times u} = i \text{ for } u \in \partial I^n\}$ [5] and the relative evaluation subgroups $G_n^{\mathrm{Rel}}(X, A)$ are defined by $\omega_\sharp(\pi_n(X^A, A^A, i)) = \{[f] \in \pi_n(X, A) \mid \exists \text{ map } H : (X \times I^n, A \times \partial I^n) \to (X, A) \text{ such that } [H|_{x_0 \times I^n}] = [f] \text{ and } H|_{X \times u} = 1_X \text{ for } u \in J^{n-1}\}$ [6]. The inclusion maps and the evaluation maps induce the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_n(A^A) \arrow[r, "i_\sharp"] \arrow[d, "\omega_\sharp"] & \pi_n(X^A) \arrow[r, "j_\sharp"] \arrow[d, "\omega_\sharp"] & \pi_n(X^A, A^A) \arrow[r, "\partial"] \arrow[d, "\omega_\sharp"] & \pi_{n-1}(A^A) \arrow[r] \arrow[d, "\omega_\sharp"] & \cdots \\
\cdots \arrow[r] & G_n(A) \arrow[r, "i_\sharp"] \arrow[d, "\cap"] & G_n(X, A) \arrow[r, "j_\sharp"] \arrow[d, "\cap"] & G_n^{\mathrm{Rel}}(X, A) \arrow[r, "\partial"] \arrow[d, "\cap"] & G_{n-1}(A) \arrow[r] \arrow[d, "\cap"] & \cdots \\
\cdots \arrow[r] & \pi_n(A) \arrow[r, "i_\sharp"] & \pi_n(X) \arrow[r, "j_\sharp"] & \pi_n(X, A) \arrow[r, "\partial"] & \pi_{n-1}(A) \arrow[r] & \cdots
\end{tikzcd}$$

Since the top and the bottom rows are exact, the middle part make sequence. We call this middle sequence the G-*sequence* of a CW-pair (X, A). This sequence is not necessarily exact [7,8]. In [6,7], it was known that if the inclusion map $i : A \to X$ has a left homotopy inverse or is null homotopic, then the G-sequence of (X, A) is exact.

A map $f : X \to Y$ is a monomorphism [4] in the category of based topological spaces and based homotopy classes of maps if, for any space Z and any two maps $u, v : Z \to X$, $f \circ u \simeq f \circ v$ implies $u \simeq v$. The monomorphicity of a map is a weaker condition than the existence of a left homotopy inverse of the map. For example, the Hopf map

$h : S^3 \to S^2$ is monomorphic but it does not have a left homotopy inverse. Ganea [1] studied monomorphicity of the Hopf fibrations, especially, he showed that $h : S^7 \to S^4$ and $h : S^{15} \to S^8$ are not monomorphic.

In this paper, we show that if the inclusion map of a *CW*-pair is monomorphic in the category of based *CW*-complexes, then the pair has an exact G-sequence. As applications of exact G-sequences, we can show that if a map from a space X to a G-space is monomorphic, then X is also a G-space. It is also shown that the Hopf map $h : S^7 \to S^4$ is not monomorphic (which was erroneously listed as monomorphic by Hilton) by a different method from Ganea.

2. G-sequences and monomorphisms

Let $p : (X, x_0) \to (Y, y_0)$ be a map in the category of based *CW*-complexes and let $\hat{p} : (X_*^X, id) \to (Y_*^X, p)$ and $\bar{p} : (X^X, id) \to (Y^X, p)$ be the induced maps given by $\hat{p}(f) = pf$ and $\bar{p}(f) = pf$, respectively, where $Y_*^X = (Y, y)^{(X,x)}$ is a subspace of the function space Y^X consisting of based maps and pf denotes composition of p and f.

In order to prove the main theorem, we need to introduce a lemma.

Lemma 2.1. *If a map $p : (X, x_0) \to (Y, y_0)$ is monomorphic in the category of based CW-complexes and based homotopy classes of maps, then the induced maps $\hat{p}$ and $\bar{p}$ are monomorphic.*

Proof. We prove only the case for $\hat{p}$. Let (Z, z_0) be a based *CW*-complex and $u, v : (Z, z_0) \to (X_*^X, id)$ be pointed maps such that $\hat{p}u$ is homotopic to $\hat{p}v$ relative to a base point z_0. Since the conjugate map

$$\mu : (Y_*^X, p)^{(Z,z_0)} \to (Y, y_0)^{((Z\times X)/(Z\times x_0), q_0)}$$

is the natural homeomorphism, $\mu(\hat{p}u)$ is homotopic to $\mu(\hat{p}v)$ as a map from $((Z \times X)/(Z \times x_0), q_0)$ to (Y, y_0), where q_0 is the base point of $(Z \times X)/(Z \times x_0)$. Since $\mu(\hat{p}u) = p\mu(u)$ and $\mu(\hat{p}v) = p\mu(v)$, we have $\mu(u)$ is homotopic to $\mu(v)$ by the fact that p is a monomorphism. This implies that u is homotopic to v (not necessarily relative to base point). Now we need to show these two maps to be homotopic relative to base point.

Let $H : Z \times I \to X_*^X$ be a homotopy from u to v and $\sigma(t) = H(z_0, t)$ be the loop at id_X in X_*^X. We define a map

$$K : (Z \times I \times 0) \cup (z_0 \times I \times I) \cup (Z \times 0 \times I) \cup (Z \times 1 \times I) \to X_*^X$$

by $K(z, s, 0) = H(z, s) \circ \sigma(1-s)$, $K(z_0, s, t) = \sigma(s(1-t)) \circ \sigma(1-s(1-t))$, $K(z, 0, t) = u(z)$ and $K(z, 1, t) = H(z, 1) \circ \sigma(1-t) \circ \sigma(t)$. By the homotopy extension property, there is an extension $\overline{K} : (Z \times I \times I) \to X_*^X$ of K. If we take a homotopy $G : Z \times I \to X_*^X$ by $G(z, s) = \overline{K}(z, s, 1)$, then G is a homotopy from u to v relative to z_0. Therefore $\hat{p}$ is a monomorphism. □

Theorem 2.2. *Let (X, A) be a connected CW-pair. If the inclusion map $i : A \to X$ is monomorphic, then the G-sequence of (X, A) is exact.*

Proof. If we consider the following commutative diagram

$$\begin{tikzcd}
(A_*^A, id) \arrow[r, "\hat{i}"] \arrow[d, "k_1"] & (X_*^A, i) \arrow[d, "k_2"] \\
(A^A, id) \arrow[r, "\bar{i}"] \arrow[d, "\omega_1"] & (X^A, i) \arrow[d, "\omega_2"] \\
A \arrow[r, "i"] & X
\end{tikzcd}$$

where k_i's are inclusions, ω_i's evaluation maps, $\bar{i}(f) = if$ and $\hat{i}(f) = if$, then the map $(\omega_1, \omega_2): \bar{i} \to i$ is a fibre map in the category of pairs with the induced map, $\hat{i}$, of the fibers A_*^A, X_*^A of ω_1, ω_2, as the fiber, because $(A_*^A, id) \xrightarrow{k_1} (A^A, id) \xrightarrow{\omega_1} A$ and $(X_*^A, i) \xrightarrow{k_2} (X^A, i) \xrightarrow{\omega_2} X$ are fibration sequences. If we use the fibre map $(\omega_1, \omega_2): \bar{i} \to i$ in the category of pairs, the sequence

$$\cdots \to \pi_n(X_*^A, A_*^A, id) \to \pi_n(X^A, A^A, id) \to \pi_n(X, A) \to \cdots$$

is exact (see [4, p. 77]).

Since the inclusion map $i: A \to X$ is monomorphic, $\hat{i}: (A_*^A, id) \to (X_*^A, i)$ and $\bar{i}: (A^A, id) \to (X^A, i)$ are also monomorphic by Lemma 2.1. Thus we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi_n(A_*^A, id) \arrow[r, "\hat{i}_\sharp"] \arrow[d, "k_{1\sharp}"] & \pi_n(X_*^A, i) \arrow[r, "\hat{j}_\sharp"] \arrow[d, "k_{2\sharp}"] & \pi_n(X_*^A, A_*^A, id) \arrow[r] \arrow[d, "(k_1, k_2)_\sharp"] & 0 \\
0 \arrow[r] & \pi_n(A^A, id) \arrow[r, "\bar{i}_\sharp"] \arrow[d, "\omega_{1\sharp}"] & \pi_n(X^A, i) \arrow[r, "\bar{j}_\sharp"] \arrow[d, "\omega_{2\sharp}"] & \pi_n(X^A, A^A, id) \arrow[r] \arrow[d, "(\omega_1, \omega_2)_\sharp"] & 0 \\
0 \arrow[r] & \pi_n(A) \arrow[r, "i_\sharp"] & \pi_n(X) \arrow[r, "j_\sharp"] & \pi_n(X, A, x_0) \arrow[r] & 0
\end{tikzcd}$$

such that each row and each column is exact. Let us consider the G-sequence of (X, A)

$$\cdots \longrightarrow G_n(A) \xrightarrow{i_\sharp} G_n(X, A) \xrightarrow{j_\sharp} G_n^{\mathrm{Rel}}(X, A) \xrightarrow{\partial} \cdots.$$

It is easy to prove exactness at $G_n(A)$. By the surjectivity of $\bar{j}_\sharp$ and the commutativity of above diagram of homotopy groups, we have

$$G_n^{\mathrm{Rel}}(X, A) = (\omega_1, \omega_2)_\sharp \bar{j}_\sharp\big(\pi_n(X^A, i)\big) = j_\sharp \omega_{2\sharp}\big(\pi_n(X^A, i)\big) = j_\sharp\big(G_n(X, A)\big)$$

and hence the G-sequence is exact at $G_n^{\mathrm{Rel}}(X, A)$ by using the triviality of ∂.

To prove the G-sequence is exact at $G_n(X, A)$, we use the above diagram of homotopy groups. Let $\alpha \in G_n(X, A)$ with $j_\sharp(\alpha) = 0$. Then there is $\beta \in \pi_n(X^A, i)$ such that $\alpha = \omega_{2\sharp}(\beta)$. Let $\gamma = \bar{j}_\sharp(\beta)$. By the commutativity of the diagram, we have $(\omega_1, \omega_2)_\sharp(\gamma) = 0$ and hence $\gamma = (k_1, k_2)_\sharp(\delta)$ for some $\delta \in \pi_n(X_*^A, A_*^A, id)$. By surjectivity of $\hat{j}_\sharp$, there exists

$\eta \in \pi_n(X_*^A, i)$ such that $\delta = \hat{j}_\sharp(\eta)$. Since $\bar{j}_\sharp(\beta - k_{2\sharp}(\eta)) = 0$, there exists $\xi \in \pi_n(A^A, id)$ such that $\beta - k_{2\sharp}(\eta) = \bar{i}_\sharp(\xi)$. Therefore we have

$$\alpha = \omega_{2\sharp}(\beta) = \omega_{2\sharp}\big(k_{2\sharp}(\eta) + \bar{i}_\sharp(\xi)\big) = \omega_{2\sharp}\bar{i}_\sharp(\xi) = i_\sharp\omega_{1\sharp}(\xi).$$

Therefore α belongs to the image of $i_\sharp$ and hence the G-sequence of (X, A) is exact. □

Corollary 2.3 [6]. *If the inclusion map $i : A \to X$ has a left homotopy inverse, then the G-sequence of (X, A) is exact.*

3. Applications

It is difficult to determine whether a map is monomorphic or not. However if we use exactness of the G-sequence, we can obtain a useful result to determine the monomorphicity of a map. A space X is a G-space if $G_n(X) = \pi_n(X)$ for each $n \geqslant 1$ [9]. Every H-space is a G-space, but the converse is not true. It is well known that the n-sphere is a G-space if and only if n is 1, 3 or 7. In general, the image of G-spaces under monomorphisms need not be G-spaces. For example, let $h : S^3 \to S^2$ be the Hoph bundle, then h is monomorphic [4] and S^3 is a G-space but S^2 is not a G-space. However we have the following.

Theorem 3.1. *Let $p : X \to Y$ be a monomorphism and Y be a G-space. Then X is also a G-space.*

Proof. Let $p : X \to Y$ be a monomorphism and M_p be the mapping cylinder of p. Then the inclusion $i : X \to M_p$ is also monomorphic. By Theorem 2.2, the G-sequence of (M_p, X) is exact. Since Y is homotopy equivalent to M_p and Y is a G-space, we have $G_n(M_p, X) = \pi_n(M_p)$. Therefore we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & G_n(X) \arrow[r, "i_\sharp"] \arrow[d, "\cap"] & G_n(M_p, X) \arrow[r, "j_\sharp"] \arrow[d, equal] & G_n^{\mathrm{Rel}}(M_p, X) \arrow[r, "\partial"] \arrow[d, "\cap"] & 0 \\
0 \arrow[r] & \pi_n(X) \arrow[r, "i_\sharp"] & \pi_n(M_p) \arrow[r, "j_\sharp"] & \pi_n(M_p, X) \arrow[r, "\partial"] & 0
\end{tikzcd}$$

It is sufficient to show $\pi_n(X) \subset G_n(X)$. Let $[\alpha]$ be an element of $\pi_n(X)$. Then $i_\sharp([\alpha])$ belongs to $G_n(M_p, X)$ and $j_\sharp(i_\sharp([\alpha])) = 0$. By exactness of G-sequence of (M_p, X), there is an element $\beta \in G_n(X)$ such that $i_\sharp([\alpha]) = i_\sharp(\beta)$. Since $i_\sharp$ is monomorphism, $[\alpha]$ belongs to $G_n(X)$. □

Corollary 3.2. *If a map from the n-sphere to a G-space is a monomorphism, then $n = 1, 3$ or 7. Especially, if $n \neq 7$, then any map from the n-sphere to S^7 is not monomorphic.*

The Hopf map $h : S^7 \to S^4$ had first been listed as monomorphic by Hilton [4, p. 169] but it was shown to be not monomorphic by Ganea [1]. In [8, p. 293], the authors showed that the G-sequence of the CW-pair (M_h, S^7) is not exact, where M_h is the mapping cylinder of the Hopf map h. Suppose the Hopf map $h : S^7 \to S^4$ is monomorphic. Then the inclusion map $i : S^7 \to M_h$ is also monomorphic. By Theorem 2.2, the G-sequence of the pair (M_h, S^7) is exact. Therefore this also yields an alternative proof of the following.

Remark 3.3. The Hopf map $h : S^7 \to S^4$ is not a monomorphism.

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